

**IMPLICATIONS BETWEEN APPROXIMATE CONVEXITY PROPERTIES
AND APPROXIMATE HERMITE–HADAMARD INEQUALITIES**

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ABSTRACT. In this paper, the connection between the functional inequalities

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \alpha_J(x-y) \quad (x, y \in D)$$

and

$$\int_0^1 f(tx + (1-t)y)\rho(t)dt \leq \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \quad (x, y \in D)$$

is investigated, where D is a convex subset of a linear space, $f : D \rightarrow \mathbb{R}$, $\alpha_H, \alpha_J : D - D \rightarrow \mathbb{R}$ are even functions, $\lambda \in [0, 1]$, and $\rho : [0, 1] \rightarrow \mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1 \rho(t)dt = 1$.

1. INTRODUCTION

Throughout this paper \mathbb{R} , \mathbb{R}_+ , \mathbb{N} and \mathbb{Z} denote the sets of real, nonnegative real, natural and integer numbers, respectively. Let X be a real linear space and $D \subset X$ be a convex set. Denote by D^* the difference set of D :

$$D^* := D - D := \{x - y \mid x, y \in D\}.$$

Of course, D^* is convex and $0 \in D^*$. It is well-known (see [9], [20], [17], and [24], [7]) that convex functions $f : D \rightarrow \mathbb{R}$ satisfy the so-called lower and upper Hermite–Hadamard inequalities

$$(1) \quad f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt \quad (x, y \in D),$$

and

$$(2) \quad \int_0^1 f(tx + (1-t)y)dt \leq \frac{f(x)+f(y)}{2} \quad (x, y \in D),$$

respectively. The converse is also known to be true (cf. [23], [24]), i.e., if a function $f : D \rightarrow \mathbb{R}$ which is continuous over the segments of D satisfies (1) or (2), then it is also convex.

More generally, it is easy to see that the ε -convexity of f (cf. [10]), i.e., the validity of

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \quad (x, y \in D, t \in [0, 1]),$$

implies the following ε -Hermite–Hadamard inequalities

$$(3) \quad f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt + \varepsilon \quad (x, y \in D).$$

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and

$$(4) \quad \int_0^1 f(tx + (1-t)y)dt \leq \frac{f(x) + f(y)}{2} + \varepsilon \quad (x, y \in D).$$

Concerning the reversed implication, Nikodem, Riedel, and Sahoo in [25] have recently shown that the ε -Hermite–Hadamard inequalities (3) and (4) do not imply the $c\varepsilon$ -convexity of f (with any $c > 0$). Thus, in order to obtain results that establish implications between the approximate Hermite–Hadamard inequalities and the approximate Jensen inequality, one has to consider these inequalities with nonconstant error terms. More precisely, we will investigate the connection between the following functional inequalities:

$$(5) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + \alpha_J(x-y) \quad (x, y \in D),$$

$$(6) \quad f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt + \alpha_H(x-y) \quad (x, y \in D),$$

$$(7) \quad \int_0^1 f(tx + (1-t)y)\rho(t)dt \leq \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \quad (x, y \in D),$$

where $\alpha_H, \alpha_J : D^* \rightarrow \mathbb{R}_+$ are given even functions, $\lambda \in [0, 1]$, and $\rho : [0, 1] \rightarrow \mathbb{R}_+$ is an integrable nonnegative function with

$$\int_0^1 \rho(t)dt = 1.$$

In order to describe the old and new results about the connection of the approximate Jensen convexity inequality (5) and the approximate lower and upper Hermite–Hadamard inequalities (6) and (7), we need to introduce the following terminology.

For a function $f : D \rightarrow \mathbb{R}$, we say that f is *lower hemicontinuous*, *upper hemicontinuous*, and *hemiintegrable* on D if, for all $x, y \in D$, the mapping

$$t \mapsto f(tx + (1-t)y) \quad (t \in [0, 1])$$

is lower semicontinuous, upper semicontinuous, and Lebesgue integrable on $[0, 1]$, respectively. We say that a function $h : D^* \rightarrow \mathbb{R}$ is *radially lower semicontinuous*, *radially upper semicontinuous*, *radially increasing*, *radially measurable*, and *radially bounded*, if for all $u \in D^*$, the mapping

$$(8) \quad t \mapsto h(tu) \quad (t \in [0, 1])$$

is lower semicontinuous, upper semicontinuous, increasing, measurable, and bounded on $[0, 1]$, respectively. (Note that, by the convexity of D^* and $0 \in D^*$, the function (8) is correctly defined.)

In [16], the relationships between the approximate lower Hermite–Hadamard inequality (6) and approximate Jensen convexity inequality (5) were examined by Háy and Páles, who obtained the following results.

Theorem A. *Let $\alpha_J : D^* \rightarrow \mathbb{R}_+$ be a radially Lebesgue integrable even function. Assume that $f : D \rightarrow \mathbb{R}$ is hemiintegrable on D and satisfies the approximate Jensen inequality (5). Then f also satisfies the approximate lower Hermite–Hadamard inequality (6), where $\alpha_H : D^* \rightarrow \mathbb{R}$ is defined by*

$$(9) \quad \alpha_H(u) := \int_0^1 \alpha_J(|1-2t|u)dt \quad (u \in D^*).$$

Theorem B. *Let $\alpha_H : D^* \rightarrow \mathbb{R}_+$ be an even function. Assume that $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality (6). Then f satisfies the approximate Jensen inequality (5) where $\alpha_J : 2D^* \rightarrow \mathbb{R}_+$ is a radially increasing nonnegative solution of the functional inequality*

$$(10) \quad \int_0^1 \alpha_J(2tu)dt + \alpha_H(u) \leq \alpha_J(u) \quad (u \in D^*).$$

The main new results of Section 2 are the following two theorems that are analogous to Theorem A above.

Theorem 1.1. *Let $\alpha_J : D^* \rightarrow \mathbb{R}$ be radially bounded, measurable and $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be a Lebesgue integrable function with $\int_0^1 \rho = 1$. Assume that $f : D \rightarrow \mathbb{R}$ is hemiintegrable and approximately Jensen convex in the sense of (5). Then f also satisfies the approximate upper Hermite–Hadamard inequality (7) with $\lambda := \int_0^1 t\rho(t)dt$ and $\alpha_H : D^* \rightarrow \mathbb{R}$ defined by*

$$(11) \quad \alpha_H(u) := \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \alpha_J(2d_{\mathbb{Z}}(2^n t)u) \rho(t)dt \quad (u \in D^*),$$

where, for $s \in \mathbb{R}$, $d_{\mathbb{Z}}(s) := \text{dist}(s, \mathbb{Z}) = \inf\{|s - k| : k \in \mathbb{Z}\}$.

Theorem 1.2. *Let $\alpha_J : D^* \rightarrow \mathbb{R}_+$ be radially increasing such that*

$$(12) \quad \sum_{n=0}^{\infty} \alpha_J\left(\frac{u}{2^n}\right) < \infty \quad (u \in D^*).$$

If $f : D \rightarrow \mathbb{R}$ is upper hemicontinuous and α_J -Jensen convex on D , i.e., (5) holds, then f also satisfies the Hermite–Hadamard inequality (7) with $\lambda := \int_0^1 t\rho(t)dt$ and $\alpha_H : D^ \rightarrow \mathbb{R}$ is defined by*

$$(13) \quad \alpha_H(u) := \sum_{n=0}^{\infty} 2\alpha_J\left(\frac{u}{2^n}\right) \int_0^1 d_{\mathbb{Z}}(2^n t)\rho(t)dt \quad (u \in D^*).$$

The main result of Section 3 is the following theorem which corresponds to Theorem B above.

Theorem 1.3. *Let $\alpha_H : D^* \rightarrow \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \geq 0$ and $p > 0$ such that*

$$(14) \quad \rho(t) \leq c(-\ln|1 - 2t|)^{p-1} \quad (t \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[),$$

and $\lambda \in [0, 1]$. Then every $f : D \rightarrow \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality (7), fulfills the approximate Jensen inequality (5) provided that $\alpha_J : D^ \rightarrow \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality*

$$(15) \quad \alpha_J(u) \geq \int_0^1 \alpha_J(|1 - 2t|u)\rho(t)dt + \alpha_H(u) \quad (u \in D^*)$$

and $\alpha_J(0) \geq \alpha_H(0)$.

In Section 2, implications from inequality (5) to (7) will be investigated.

A weaker form of Theorem 1.1 could be deduced from the following result which was obtained by the authors in [18].

Theorem C. *Let $\alpha_J : D^* \rightarrow \mathbb{R}_+$ be radially bounded and even. Then, an upper hemicontinuous function $f : D \rightarrow \mathbb{R}$ is α_J -Jensen convex on D , i.e., (5) holds if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \sum_{n=0}^{\infty} \frac{\alpha_J(2d_{\mathbb{Z}}(2^n t)(x-y))}{2^n} \quad (x, y \in D, t \in [0, 1]). \quad (16)$$

In the proof of Theorem 1.1, we will directly derive (7) from (5) under more general circumstances.

To deduce Theorem 1.2, the following result of Jacek and Józef Tabor [27] will be used.

Theorem D. *Let $\alpha_J : D^* \rightarrow \mathbb{R}_+$ be radially increasing and even such that, for all $u \in D^*$, $\sum_{n=0}^{\infty} \alpha_J(2^{-n}u) < \infty$. Then, an upper hemicontinuous function $f : D \rightarrow \mathbb{R}$ is α_J -Jensen convex on D , i.e., (5) holds if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \sum_{n=0}^{\infty} 2\alpha_J\left(\frac{x-y}{2^n}\right)d_{\mathbb{Z}}(2^n t) \quad (x, y \in D, t \in [0, 1]). \quad (17)$$

In the particular case when α_J is a linear combination of power functions, we also deduce some consequences of Theorem 1.1 and Theorem 1.2. For this aim, we will have to recall two different notions of Takagi type functions. For $q > 0$, define the functions $T_q : \mathbb{R} \rightarrow \mathbb{R}$ and $S_q : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_q(x) := \sum_{n=0}^{\infty} \frac{(2d_{\mathbb{Z}}(2^n x))^q}{2^n}, \quad S_q(x) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n x)}{2^{nq-1}} \quad (x \in \mathbb{R}). \quad (18)$$

They generalize the classical Takagi function $T_1 = S_1 = T$ in two ways. It is more difficult to see that $T_2 = S_2$ is also valid. These functions have an important role in approximate convex analysis.

The importance of the functions T_q introduced above is enlightened by the following result (cf. [14], [15], [11], [12]) which is a generalization of the celebrated Bernstein–Doetsch theorem [2].

Theorem E. *Let X be a normed space, $q > 0$ and $a \geq 0$. Then a locally upper bounded function $f : D \rightarrow \mathbb{R}$ is (a, q) -Jensen convex on D , i.e.,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + a\|x-y\|^q \quad (x, y \in D),$$

if and only

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + aT_q(t)\|x-y\|^q \quad (x, y \in D, t \in [0, 1]). \quad (19)$$

The other Takagi type function S_q was introduced by Tabor and Tabor. Its role and importance in the theory of approximate convexity is shown by the next theorem ([27], [28]).

Theorem F. *Let X be a normed space, $q > 0$ and $a \geq 0$. Then a locally upper bounded function $f : D \rightarrow \mathbb{R}$ is (a, q) -Jensen convex on D , i.e.,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + a\|x-y\|^q \quad (x, y \in D),$$

if and only if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + aS_q(t)\|x-y\|^q \quad (x, y \in D, t \in [0, 1]). \quad (20)$$

In view of the results in the papers [18] and [19], the error terms in (19) and (20) are the best possible if $0 < q \leq 1$ and $1 \leq q \leq 2$, respectively.

In Section 3, for every parameter $p > 0$, we define a class of functions, denoted by Φ_p . A certain convolution-like operation is also introduced in $\bigcup_{p>0} \Phi_p$ and its properties are described in Propositions 3.1–3.5. These tools will be instrumental in the proof of Theorem 1.3 which will be carried out in several steps. Finally, when the error function α_H is a linear combination of power functions, we will also deduce some corollaries of Theorem 1.3.

2. FROM JENSEN INEQUALITY TO HERMITE-HADAMARD INEQUALITY

The following statement will be essential to obtain our first main result, Theorem 1.1.

Proposition 2.1. *Let $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be a Lebesgue integrable function. Then, the function $\psi : [0, 1] \rightarrow \mathbb{R}$ defined by*

$$(21) \quad \psi(t) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} \left(\sum_{k=0}^{2^n-1} \rho\left(\frac{t+k}{2^n}\right) \right) \quad (t \in [0, 1]),$$

is a nonnegative integrable solution of the functional equation

$$(22) \quad \rho(t) = 2\psi(t) - \frac{\psi(\frac{t}{2}) + \psi(\frac{t+1}{2})}{2} \quad (t \in [0, 1]).$$

Furthermore,

$$(23) \quad \int_0^1 \psi = \int_0^1 \rho, \quad \int_{\frac{1}{2}}^1 \psi = \int_0^1 t\rho(t)dt \quad \text{and} \quad \int_0^{\frac{1}{2}} \psi = \int_0^1 (1-t)\rho(t)dt.$$

Proof. Define the sequence $\psi_n : [0, 1] \rightarrow \mathbb{R}$, by

$$(24) \quad \psi_0 := \frac{1}{2}\rho, \quad \psi_n(t) := \frac{1}{2}\rho(t) + \frac{1}{4}(\psi_{n-1}(\frac{t}{2}) + \psi_{n-1}(\frac{t+1}{2})) \quad (t \in [0, 1], n \in \mathbb{N}).$$

Then the sequence (ψ_n) is nondecreasing, i.e.,

$$(25) \quad 0 \leq \psi_{n-1} \leq \psi_n, \quad \text{and} \quad \int_0^1 \psi_n = \frac{2^{n+1} - 1}{2^{n+1}} \int_0^1 \rho \quad (n \in \mathbb{N}).$$

We prove (25) by induction on $n \in \mathbb{N}$. For $n = 1$, by the definition of ψ_1 and the nonnegativity of ρ , for $t \in [0, 1]$, we have that

$$\psi_1(t) = \frac{1}{2}\rho(t) + \frac{1}{4}(\psi_0(\frac{t}{2}) + \psi_0(\frac{t+1}{2})) = \frac{1}{2}\rho(t) + \frac{1}{8}(\rho(\frac{t}{2}) + \rho(\frac{t+1}{2})) \geq \frac{1}{2}\rho(t) = \psi_0(t),$$

and

$$\int_0^1 \psi_0 = \frac{1}{2} \int_0^1 \rho.$$

Assume that, for some $n \in \mathbb{N}$, (25) holds and consider the case $n + 1$. By the definition of ψ_{n+1} , the inductive assumption and the nonnegativity of ψ_n , for $t \in [0, 1]$, yields

$$\psi_{n+1}(t) = \frac{1}{2}\rho(t) + \frac{1}{4}(\psi_n(\frac{t}{2}) + \psi_n(\frac{t+1}{2})) \geq \frac{1}{2}\rho(t) + \frac{1}{4}(\psi_{n-1}(\frac{t}{2}) + \psi_{n-1}(\frac{t+1}{2})) = \psi_n(t).$$

Using the definition ψ_{n+1} , the substitution $s := \frac{t}{2}$ and $s := \frac{t+1}{2}$, finally the inductive assumption, we get

$$\begin{aligned}\int_0^1 \psi_{n+1} &= \frac{1}{2} \int_0^1 \rho + \frac{1}{4} \left(\int_0^1 \psi_n\left(\frac{t}{2}\right) dt + \int_0^1 \psi_n\left(\frac{t+1}{2}\right) dt \right) = \frac{1}{2} \int_0^1 \rho + \frac{1}{2} \left(\int_0^{\frac{1}{2}} \psi_n + \int_{\frac{1}{2}}^1 \psi_n \right) \\ &= \frac{1}{2} \int_0^1 \rho + \frac{1}{2} \int_0^1 \psi_n = \left(\frac{1}{2} + \frac{2^{n+1} - 1}{2^{n+2}} \right) \int_0^1 \rho = \frac{2^{n+2} - 1}{2^{n+2}} \int_0^1 \rho.\end{aligned}$$

Denote by $L^1[0, 1]$ the space of Lebesgue integrable functions $\chi : [0, 1] \rightarrow \mathbb{R}$. Then $L^1[0, 1]$ is a Banach-space with the standard norm $\|\chi\|_1 := \int_0^1 |\chi|$. Now we prove that (ψ_n) is a Cauchy sequence in $L^1[0, 1]$. Using (25), for $n \leq m$, we get that

$$\|\psi_m - \psi_n\|_1 = \int_0^1 (\psi_m - \psi_n) = \frac{2^m - 2^n}{2^{n+m+1}} \int_0^1 \rho \leq \frac{1}{2^n} \int_0^1 \rho,$$

which implies that (ψ_n) is indeed a Cauchy sequence. Hence it converges to a function $\psi \in L^1[0, 1]$. To prove (21), we show, by induction on $n \in \mathbb{N}$, that

$$(26) \quad \psi_n(t) = \frac{1}{2} \sum_{i=0}^n \frac{1}{4^i} \left(\sum_{k=0}^{2^i-1} \rho\left(\frac{t+k}{2^i}\right) \right) \quad (t \in [0, 1])$$

holds. For $n = 0$, we have an obvious identity. Assume that (26) holds some $n \in \mathbb{N}$. Using the definition of ψ_{n+1} and the inductive assumption, we obtain

$$\begin{aligned}\psi_{n+1}(t) &= \frac{1}{2} \rho(t) + \frac{1}{4} (\psi_n\left(\frac{t}{2}\right) + \psi_n\left(\frac{t+1}{2}\right)) \\ &= \frac{1}{2} \rho(t) + \frac{1}{8} \sum_{i=0}^n \frac{1}{4^i} \left(\sum_{k=0}^{2^i-1} \rho\left(\frac{t+2k}{2^{i+1}}\right) \right) + \frac{1}{8} \sum_{i=0}^n \frac{1}{4^i} \left(\sum_{k=0}^{2^i-1} \rho\left(\frac{t+2k+1}{2^{i+1}}\right) \right) \\ &= \frac{1}{2} \rho(t) + \frac{1}{2} \sum_{i=0}^n \frac{1}{4^{i+1}} \sum_{k=0}^{2^i-1} \left(\rho\left(\frac{t+2k}{2^{i+1}}\right) + \rho\left(\frac{t+2k+1}{2^{i+1}}\right) \right) \\ &= \frac{1}{2} \rho(t) + \frac{1}{2} \sum_{i=0}^n \frac{1}{4^{i+1}} \sum_{k=0}^{2^{i+1}-1} \rho\left(\frac{t+k}{2^{i+1}}\right) = \frac{1}{2} \sum_{i=0}^{n+1} \frac{1}{4^i} \left(\sum_{k=0}^{2^i-1} \rho\left(\frac{t+k}{2^i}\right) \right),\end{aligned}$$

which proves (26). Thus, taking the limit $n \rightarrow \infty$ in (26), we obtain (21).

To prove the first expression in (23), integrate (22) on $[0, 1]$, then we get

$$\int_0^1 \rho = 2 \int_0^1 \psi - \frac{\int_0^1 \psi\left(\frac{t}{2}\right) dt + \int_0^1 \psi\left(\frac{t+1}{2}\right) dt}{2} = 2 \int_0^1 \psi - \left(\int_0^{\frac{1}{2}} \psi + \int_{\frac{1}{2}}^1 \psi \right) = \int_0^1 \psi.$$

To prove the second expression in (23), multiply (22) by t and integrate it on $[0, 1]$. Thus we get

$$\begin{aligned}\int_0^1 t \rho(t) dt &= 2 \int_0^1 t \psi(t) dt - \left(\int_0^1 \frac{t}{2} \psi\left(\frac{t}{2}\right) dt + \int_0^1 \frac{t+1}{2} \psi\left(\frac{t+1}{2}\right) dt \right) + \frac{1}{2} \int_0^1 \psi\left(\frac{t+1}{2}\right) dt \\ &= 2 \int_0^1 t \psi(t) dt - 2 \left(\int_0^{\frac{1}{2}} s \psi(s) ds + \int_{\frac{1}{2}}^1 s \psi(s) ds \right) + \int_{\frac{1}{2}}^1 \psi = \int_{\frac{1}{2}}^1 \psi.\end{aligned}$$

The last equality in (23) is a consequence of the first and second equalities. \square

Proof of Theorem 1.1. By Proposition 2.1, the function $\psi : [0, 1] \rightarrow \mathbb{R}_+$ defined by (21) is a Lebesgue integrable function satisfying the functional equation (22) for which (23) holds.

Let $f : D \rightarrow \mathbb{R}$ be an approximately Jensen convex function. Let $x, y \in D$ be arbitrary fixed. Then, by approximate Jensen convexity of f , we have that

$$f(tx + (1-t)y) \leq \begin{cases} \frac{f(2tx + (1-2t)y) + f(y)}{2} + \alpha_J(2t(x-y)) & (t \in [0, \frac{1}{2}]), \\ \frac{f(x) + f((2t-1)x + (2-2t)y)}{2} + \alpha_J((2-2t)(x-y)) & (t \in [\frac{1}{2}, 1]). \end{cases}$$

Multiplying the above inequality by $2\psi(t)$, taking the integral over $[0, 1]$, we get

$$(27) \quad \int_0^1 f(tx + (1-t)y)2\psi(t)dt \leq \int_0^{\frac{1}{2}} \left(f(2tx + (1-2t)y) + f(y) + 2\alpha_J(2t(x-y)) \right) \psi(t)dt \\ + \int_{\frac{1}{2}}^1 \left(f(x) + f((2t-1)x + (2-2t)y) + 2\alpha_J((2-2t)(x-y)) \right) \psi(t)dt.$$

Substituting $t := \frac{s}{2}$ and $t := \frac{1+s}{2}$ in the first and second terms on the right hand side of (27), using (23), and observing that $2d_{\mathbb{Z}}(t) = \min(2t, 2-2t)$, we have that

$$(28) \quad \int_0^{\frac{1}{2}} \left(f(2tx + (1-2t)y) + f(y) + 2\alpha_J(2t(x-y)) \right) \psi(t)dt \\ = f(y) \int_0^1 (1-t)\rho(t)dt + \frac{1}{2} \int_0^1 f(sx + (1-s)y)\psi(\frac{s}{2})ds + 2 \int_0^{\frac{1}{2}} \alpha_J(2d_{\mathbb{Z}}(t)(x-y))\psi(t)dt, \\ \int_{\frac{1}{2}}^1 \left(f(x) + f((2t-1)x + (2-2t)y) + 2\alpha_J((2-2t)(x-y)) \right) \psi(t)dt \\ = f(x) \int_0^1 t\rho(t)dt + \frac{1}{2} \int_0^1 f(sx + (1-s)y)\psi(\frac{1+s}{2})ds + 2 \int_{\frac{1}{2}}^1 \alpha_J(2d_{\mathbb{Z}}(t)(x-y))\psi(t)dt.$$

Combining (27) and (28), we get that

$$\int_0^1 f(tx + (1-t)y)\rho(t)dt = \int_0^1 f(tx + (1-t)y) \left(2\psi(t) - \frac{1}{2}\psi(\frac{t}{2}) - \frac{1}{2}\psi(\frac{t+1}{2}) \right) dt \\ \leq f(x) \int_0^1 t\rho(t)dt + f(y) \int_0^1 (1-t)\rho(t)dt + 2 \int_0^1 \alpha_J(2d_{\mathbb{Z}}(t)(x-y))\psi(t)dt.$$

To complete the proof, it remains to show that the last term containing ψ equals $\alpha_H(x - y)$. Indeed, applying formula (21) and the 1-periodicity of the function $d_{\mathbb{Z}}$, we get

$$\begin{aligned}
2 \int_0^1 \alpha_J(2d_{\mathbb{Z}}(t)(x - y))\psi(t)dt &= 2 \int_0^1 \alpha_J(2d_{\mathbb{Z}}(t)(x - y)) \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} \left(\sum_{k=0}^{2^n-1} \rho\left(\frac{t+k}{2^n}\right) dt \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{4^n} \left(\sum_{k=0}^{2^n-1} \int_0^1 \alpha_J(2d_{\mathbb{Z}}(t)(x - y)) \rho\left(\frac{t+k}{2^n}\right) dt \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{4^n} \left(\sum_{k=0}^{2^n-1} 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \alpha_J(2d_{\mathbb{Z}}(2^n s - k)(x - y)) \rho(s) ds \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \alpha_J(2d_{\mathbb{Z}}(2^n s)(x - y)) \rho(s) ds = \alpha_H(x - y),
\end{aligned}$$

which proves the statement. \square

The other form of the error function α_H stated in Theorem 1.2 can be obtained by using Theorem D by Jacek Tabor and Józef Tabor [27].

Proof of Theorem 1.2. If $f : D \rightarrow \mathbb{R}$ is upper hemicontinuous and α_J -Jensen convex on D , then (17) holds. Multiplying this inequality by $\rho(t)$ and then integrating with respect to t over $[0, 1]$, (13) follows immediately. \square

Let X be a normed space. Next, we consider the case, when α_J is a linear combination of the powers of the norm with positive exponents, i.e., if α_J is of the form

$$(29) \quad \alpha_J(u) := \int_{]0, \infty[} \|u\|^q d\mu_J(q) \quad (u \in D^*),$$

where μ_J is a nonnegative Borel measure on the interval $]0, \infty[$. An important particular case is when μ_J is of the form $\sum_{i=1}^k c_i \delta_{q_i}$, where $c_i \in \mathbb{R}_+$, $q_i > 0$ and δ_{q_i} stands for the Dirac measure concentrated at q_i for $i \in \{1, \dots, k\}$.

Theorem 2.2. *Let $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be a Lebesgue integrable function with $\int_0^1 \rho = 1$ and let μ_J be a signed Borel measure on $]0, \infty[$ such that*

$$\int_{]0, \infty[} \|u\|^q d|\mu_J|(q) < \infty \quad (u \in D^*).$$

Assume that $f : D \rightarrow \mathbb{R}$ is hemiintegrable on D and is approximately Jensen convex in the following sense

$$(30) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \int_{]0, \infty[} \|x-y\|^q d\mu_J(q) \quad (x, y \in D).$$

Then f also satisfies the approximate Hermite–Hadamard inequality

$$(31) \quad \int_0^1 f(tx + (1-t)y) \rho(t) dt \leq \lambda f(x) + (1-\lambda)f(y) + \int_{]0, \infty[} \int_0^1 T_q(t) \rho(t) dt \|x-y\|^q d\mu_J(q) \quad (x, y \in D),$$

with $\lambda := \int_0^1 t \rho(t) dt$.

Proof. It is easy to see that α_J defined by (29) is radially bounded and measurable. Thus, by Theorem 1.1, it is enough to compute the error function α_H defined by (11). Hence, using (11), (29), Fubini's theorem and Lebesgue's theorem, we obtain

$$\begin{aligned}\alpha_H(u) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \int_{]0,\infty[} \|(2d_{\mathbb{Z}}(2^n t)u)\|^q d\mu_J(q) \rho(t) dt \\ &= \int_{]0,\infty[} \int_0^1 \sum_{n=0}^{\infty} \frac{(2d_{\mathbb{Z}}(2^n t))^q}{2^n} \rho(t) dt \|u\|^q d\mu_J(q) = \int_{]0,\infty[} \int_0^1 T_q(t) \rho(t) dt \|u\|^q d\mu_J(q) \quad (u \in D^*),\end{aligned}$$

which completes the proof. \square

Theorem 2.3. *Let $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be a Lebesgue integrable function with $\int_0^1 \rho = 1$ and let μ_J be a nonnegative Borel measure on $]0, \infty[$, such that*

$$(32) \quad \int_{]0,\infty[} \|u\|^q d\mu_J(q) < \infty \quad (u \in D^*)$$

and

$$(33) \quad \int_{]0,\infty[} \frac{2^q}{2^q - 1} d\mu_J(q) < \infty.$$

Assume that $f : D \rightarrow \mathbb{R}$ is upper hemicontinuous and approximately Jensen convex in the sense of (30). Then f also satisfies the following approximate Hermite-Hadamard inequality

$$(34) \quad \int_0^1 f(tx + (1-t)y) \rho(t) dt \leq \lambda f(x) + (1-\lambda)f(y) + \int_{]0,\infty[} \int_0^1 S_q(t) \rho(t) dt \|x - y\|^q d\mu_J(q) \quad (x, y \in D),$$

with $\lambda := \int_0^1 t \rho(t) dt$.

Proof. Consider the function α_J defined by (29). Then, for all $u \in D^*$, the mapping $t \mapsto \alpha_J(tu) = \int_{]0,\infty[} t^q \|u\|^q d\mu_J(q)$ is increasing on $[0, 1]$ and, for all $u \in D^*$,

$$\sum_{n=0}^{\infty} \alpha_J\left(\frac{u}{2^n}\right) = \sum_{n=0}^{\infty} \int_{]0,\infty[} \|2^{-n}u\|^q d\mu_J(q) = \int_{]0,\infty[} \frac{2^q}{2^q - 1} \|u\|^q d\mu_J(q).$$

If $\|u\| \leq 1$, then the latter series is convergent in virtue of (33). For $\|u\| > 1$, we have

$$\int_{]0,\infty[} \frac{2^q}{2^q - 1} \|u\|^q d\mu_J(q) \leq \|u\| \int_{]0,1[} \frac{2^q}{2^q - 1} d\mu_J(q) + 2 \int_{]1,\infty[} \|u\|^q d\mu_J(q) < \infty,$$

which proves the convergence condition (12). Thus, by Theorem 1.2, it is enough to compute the error function α_H defined by (13). Hence, using (13), (29), Fubini's theorem and Lebesgue's

theorem, we obtain

$$\begin{aligned}\alpha_H(u) &= \sum_{n=0}^{\infty} 2 \int_{]0, \infty[} \left(\frac{\|u\|}{2^n} \right)^q d\mu_J(q) \int_0^1 d_{\mathbb{Z}}(2^n t) \rho(t) dt \\ &= \int_{]0, \infty[} \int_0^1 \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq-1}} \rho(t) dt \|u\|^q d\mu_J(q) = \int_{]0, \infty[} \int_0^1 S_q(t) \rho(t) dt \|u\|^q d\mu_J(q) \quad (u \in D^*),\end{aligned}$$

which completes the proof. \square

Now we consider the case in the previous theorems when $\rho \equiv 1$ and the measure μ_J is the Dirac measure $a\delta_q$.

Corollary 2.4. *Let $a \in \mathbb{R}_+$ and $q > 0$. Assume that $f : D \rightarrow \mathbb{R}$ is hemiintegrable and satisfies the following approximate Jensen convexity inequality*

$$(35) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + a\|x-y\|^q \quad (x, y \in D).$$

Then f also satisfies the following approximate Hermite–Hadamard inequality

$$(36) \quad \int_0^1 f(tx + (1-t)y) dt \leq \frac{f(x)+f(y)}{2} + \frac{2a}{q+1} \|x-y\|^q \quad (x, y \in D).$$

Proof. The conditions of Theorem 2.2 hold with $\rho \equiv 1$ and $\mu_J := a\delta_q$. Then (30) holds by (35), hence to prove the statement, it is enough to compute the error term in (31). Using the definition of the T_q , the substitution $s := 2^n t$ and the 1-periodicity of $d_{\mathbb{Z}}^q$, we get

$$\begin{aligned}\int_0^1 T_q(t) dt &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(2d_{\mathbb{Z}}(2^n t))^q}{2^n} \right) dt = \sum_{n=0}^{\infty} \frac{2^q}{2^n} \int_0^1 (d_{\mathbb{Z}}(2^n t))^q dt = \sum_{n=0}^{\infty} \frac{2^q}{2^{2n}} \int_0^{2^n} (d_{\mathbb{Z}}(s))^q ds \\ &= \sum_{n=0}^{\infty} \frac{2^q}{2^n} \int_0^1 (d_{\mathbb{Z}}(s))^q ds = \sum_{n=0}^{\infty} \frac{2^q}{2^n} \left(\int_0^{\frac{1}{2}} s^q ds + \int_{\frac{1}{2}}^1 (1-s)^q ds \right) \\ &= \sum_{n=0}^{\infty} \frac{2^q}{2^n} \left(\frac{(\frac{1}{2})^{q+1}}{q+1} + \frac{(\frac{1}{2})^{q+1}}{q+1} \right) = \frac{1}{q+1} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{2}{q+1} \quad (u \in D^*).\end{aligned}$$

Thus, (31) reduces to (36), which proves the statement. \square

Corollary 2.5. *Let $a \in \mathbb{R}_+$ and $q > 0$. Assume that $f : D \rightarrow \mathbb{R}$ is upper hemicontinuous and satisfies the approximate Jensen convexity inequality (35). Then f also satisfies the following approximate Hermite–Hadamard inequality*

$$(37) \quad \int_0^1 f(tx + (1-t)y) dt \leq \frac{f(x)+f(y)}{2} + \frac{2^q a}{2^{q+1}-2} \|x-y\|^q \quad (x, y \in D).$$

Proof. The conditions of Theorem 2.3 are satisfied with $\rho \equiv 1$ and $\mu_J := a\delta_q$. Then (30) holds by (35), hence to prove the statement, it is enough to compute the error term in (34). Using the

definition of the S_q , the substitution $s := 2^n t$ and the 1-periodicity of $d_{\mathbb{Z}}$, we get

$$\begin{aligned} \int_0^1 S_q(t) dt &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq-1}} \right) dt = \sum_{n=0}^{\infty} \frac{1}{2^{nq-1}} \int_0^1 d_{\mathbb{Z}}(2^n t) dt = \sum_{n=0}^{\infty} \frac{1}{2^{nq+n-1}} \int_0^{2^n} d_{\mathbb{Z}}(s) ds \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{nq-1}} \int_0^1 d_{\mathbb{Z}}(s) ds = \sum_{n=0}^{\infty} \frac{1}{2^{nq-1}} \left(\int_0^{\frac{1}{2}} s ds + \int_{\frac{1}{2}}^1 (1-s) ds \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{nq-1}} \frac{1}{4} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^{nq}} = \frac{2^q}{2^{q+1} - 2} \quad (u \in D^*). \end{aligned}$$

Thus, (34) reduces to (37), which completes the proof. \square

Remark. The constants in the two error terms obtained in (36) and (37) are comparable in the following way: for $q \in]0, 1[\cup]2, \infty[$,

$$\frac{2}{q+1} < \frac{2^q}{2^{q+1} - 2}$$

and the inequality reverses for $q \in]1, 2[$.

3. FROM HERMITE-HADAMARD INEQUALITY TO JENSEN INEQUALITY

For $p > 0$, define the class of functions Φ_p by

$$\Phi_p := \left\{ \varphi :]0, 1[\rightarrow \mathbb{R} \mid \varphi \text{ is Lebesgue measurable and } \|\varphi\|_p := \sup_{t \in]0, 1[} |\ln t|^{1-p} |\varphi(t)| < \infty \right\}.$$

In the sequel, Γ denotes Euler's Gamma function.

Proposition 3.1. *For all $p > 0$, the elements of Φ_p are Lebesgue integrable functions and*

$$(38) \quad \|\varphi\|_1 = \int_0^1 |\varphi(t)| dt \leq \Gamma(p) \|\varphi\|_p \quad (\varphi \in \Phi_p).$$

Proof. Let $p > 0$ and $\varphi \in \Phi_p$. From the definition of Φ_p , we get that

$$|\varphi(t)| \leq \|\varphi\|_p (-\ln t)^{p-1} \quad (t \in]0, 1[).$$

Thus, with the substitution $s = -\ln t$, we get

$$\int_0^1 |\varphi(t)| dt \leq \|\varphi\|_p \int_0^1 (-\ln t)^{p-1} dt = \|\varphi\|_p \int_0^{\infty} s^{p-1} e^{-s} ds = \Gamma(p) \|\varphi\|_p < \infty,$$

which proves the integrability of φ and (38). \square

Proposition 3.2. *For $p, q > 0$ and $\varphi \in \Phi_p$, $\psi \in \Phi_q$, the function $\varphi * \psi$ defined by*

$$(\varphi * \psi)(t) := \int_t^1 \frac{1}{\tau} \varphi\left(\frac{t}{\tau}\right) \psi(\tau) d\tau \quad (t \in]0, 1[).$$

is continuous on the open interval $]0, 1[$, belongs to Φ_{p+q} and

$$(39) \quad \|\varphi * \psi\|_{p+q} \leq \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \|\varphi\|_p \|\psi\|_q.$$

Furthermore,

$$(40) \quad \int_0^1 (\varphi * \psi) = \int_0^1 \varphi \int_0^1 \psi.$$

Proof. Given $p, q > 0$, it is well known that the function

$$(41) \quad \tau \mapsto (1 - \tau)^{p-1} \tau^{q-1}, \quad (\tau \in]0, 1[)$$

is integrable over $[0, 1]$ and

$$B(p, q) := \int_0^1 (1 - \tau)^{p-1} \tau^{q-1} d\tau = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

By the inclusions $\varphi \in \Phi_p$, $\psi \in \Phi_q$, we have that

$$(42) \quad |\varphi(t)| \leq \|\varphi\|_p (-\ln t)^{p-1} \quad \text{and} \quad |\psi(t)| \leq \|\psi\|_q (-\ln t)^{q-1} \quad (t \in]0, 1[).$$

To prove that $\varphi * \psi$ is continuous at $t \in]0, 1[$, let $\varepsilon > 0$. By the integrability of (41), there exists $\rho \in]0, 1[$ such that, for every measurable subset $T \subseteq [0, 1]$ with $\text{meas}(T) < \rho$,

$$(43) \quad \int_T (1 - \tau)^{p-1} \tau^{q-1} d\tau < \frac{\varepsilon}{6\|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1} + 1}.$$

Define the function $\tilde{\varphi} :]0, \infty[\rightarrow \mathbb{R}$ by

$$\tilde{\varphi}(x) := \varphi(e^{-x}).$$

Then, by the first inequality in (42), we have that

$$x^{1-p} |\tilde{\varphi}(x)| \leq \|\varphi\|_p \quad (x \in]0, \infty[).$$

Applying Luzin's theorem for the bounded measurable function $x \mapsto x^{1-p} \tilde{\varphi}(x)$, we can construct a measurable set $\tilde{H} \subseteq]0, \infty[$ and a continuous function $\tilde{f} :]0, \infty[\rightarrow \mathbb{R}$ such that

$$(44) \quad \text{meas}(]0, \infty[\setminus \tilde{H}) < \frac{(-\ln t)\rho}{2}, \quad \tilde{f}|_{\tilde{H}} = \tilde{\varphi}|_{\tilde{H}} \quad \text{and} \quad |\tilde{f}(x)| \leq \|\varphi\|_p x^{p-1} \quad (x \in]0, \infty[).$$

Now define $f :]0, 1[\rightarrow \mathbb{R}$ and $H \subseteq]0, 1[$ by

$$f(t) := \tilde{f}(-\ln t) \quad (t \in]0, 1[) \quad \text{and} \quad H := \exp(-\tilde{H}).$$

In a view of (44), we get that

$$(45) \quad f|_H = \varphi|_H \quad \text{and} \quad |f(t)| \leq \|\varphi\|_p (-\ln t)^{p-1} \quad (t \in]0, 1[).$$

By the continuity of the logarithmic function, there exists $\delta \in]0, \min(t, 1-t)[$, such that, for all $s \in]t - \delta, t + \delta[$,

$$(46) \quad \|\varphi\|_p \|\psi\|_q (-\ln(s))^{p+q-1} < \|\varphi\|_p \|\psi\|_q (-\ln(t))^{p+q-1} + \frac{1}{6} \quad \text{and} \quad \left| 1 - \frac{\ln t}{\ln s} \right| < \rho.$$

For $s \in]0, 1[$, we have

$$(47) \quad \begin{aligned} |(\varphi * \psi)(t) - (\varphi * \psi)(s)| &= \left| \int_t^1 \frac{1}{\tau} \varphi\left(\frac{t}{\tau}\right) \psi(\tau) d\tau - \int_s^1 \frac{1}{\tau} \varphi\left(\frac{s}{\tau}\right) \psi(\tau) d\tau \right| \\ &\leq \begin{cases} \int_s^t \frac{1}{\tau} \left| \varphi\left(\frac{s}{\tau}\right) \right| |\psi(\tau)| d\tau + \int_t^1 \frac{1}{\tau} \left| \varphi\left(\frac{t}{\tau}\right) - \varphi\left(\frac{s}{\tau}\right) \right| |\psi(\tau)| d\tau & \text{if } s < t, \\ \int_t^s \frac{1}{\tau} \left| \varphi\left(\frac{t}{\tau}\right) \right| |\psi(\tau)| d\tau + \int_s^1 \frac{1}{\tau} \left| \varphi\left(\frac{t}{\tau}\right) - \varphi\left(\frac{s}{\tau}\right) \right| |\psi(\tau)| d\tau & \text{if } t < s. \end{cases} \end{aligned}$$

Consider first the case $s \in]t - \delta, t[$. The second inequality in (46) implies that the measure of the interval $T :=]\frac{\ln t}{\ln s}, 1[$ is smaller than ρ . Thus, inequality (43) holds with this set T . Therefore,

for the first term on the right hand side of (47), using the estimates (42), substituting $\tau = s^\sigma$, and using the first inequality in (46), we get

$$\begin{aligned}
 \int_s^t \frac{1}{\tau} |\varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau &\leq \|\varphi\|_p \|\psi\|_q \int_s^t \frac{1}{\tau} (\ln \tau - \ln s)^{p-1} (-\ln \tau)^{q-1} d\tau \\
 (48) \quad &\leq \|\varphi\|_p \|\psi\|_q (-\ln s)^{p+q-1} \int_{\frac{\ln t}{\ln s}}^1 (1-\sigma)^{p-1} \sigma^{q-1} d\sigma \\
 &< \frac{\|\varphi\|_p \|\psi\|_q (-\ln s)^{p+q-1} \varepsilon}{6 \|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1} + 1} < \frac{\varepsilon}{6}.
 \end{aligned}$$

To obtain an estimate for the second term on the right hand side of (47) (when $s < t$), we use $\varphi(x) = f(x)$ for $x \in H$ and obtain

$$\begin{aligned}
 (49) \quad \int_t^1 \frac{1}{\tau} |\varphi(\frac{t}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau &\leq \int_t^1 \frac{1}{\tau} \left(|\varphi(\frac{t}{\tau}) - f(\frac{t}{\tau})| + |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| + |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| \right) |\psi(\tau)| d\tau \\
 &\leq \int_{]t,1[\setminus tH^{-1}} \frac{1}{\tau} |\varphi(\frac{t}{\tau}) - f(\frac{t}{\tau})| |\psi(\tau)| d\tau + \int_{]t,1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau + \int_{]t,1[\setminus sH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau.
 \end{aligned}$$

The first inequality in (44) and the second estimate in (46) imply that, for $s \in]t - \delta, t]$,

$$(50) \quad \text{meas}([0, 1[\setminus (1 + (\ln s)^{-1} \tilde{H})) = (-\ln s)^{-1} \text{meas}([0, -\ln s[\setminus \tilde{H})) < \frac{\rho \ln t}{2 \ln s} < \frac{\rho(1 + \rho)}{2} < \rho.$$

Thus (43) holds with $T :=]0, 1[\setminus (1 + (\ln s)^{-1} \tilde{H})$. Using (42) and (45), then substituting $\tau = s^\sigma$ and finally applying inequality (43), we get

$$\begin{aligned}
 \int_{]t,1[\setminus sH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau &\leq \int_{]s,1[\setminus sH^{-1}} \frac{1}{\tau} (|f(\frac{s}{\tau})| + |\varphi(\frac{s}{\tau})|) |\psi(\tau)| d\tau \\
 (51) \quad &\leq 2 \|\varphi\|_p \|\psi\|_q \int_{]s,1[\setminus sH^{-1}} \frac{1}{\tau} (\ln \tau - \ln s)^{p-1} (-\ln \tau)^{q-1} d\tau \\
 &\leq 2 \|\varphi\|_p \|\psi\|_q (-\ln s)^{p+q-1} \int_{]0,1[\setminus (1 + (\ln s)^{-1} \tilde{H})) (1-\sigma)^{p-1} \sigma^{q-1} d\sigma \\
 &\leq \frac{2 \|\varphi\|_p \|\psi\|_q (-\ln s)^{p+q-1} \varepsilon}{6 \|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1} + 1} < \frac{\varepsilon}{3}.
 \end{aligned}$$

Applying this inequality for $s = t$, we also get

$$(52) \quad \int_{]t,1[\setminus tH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{3}.$$

Consider the second expression on the right hand side of (49). We prove that

$$(53) \quad \lim_{s \rightarrow t-0} \int_{]t,1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau = 0.$$

By the continuity of f , the integrand pointwise converges to zero hence, in view of Lebesgue's dominated convergence theorem, it suffices to show that the integrand admits an integrable majorant which is independent of $s \in]t - \delta, t[$.

Using the inequality (42) and (45), we get that, for all $\tau \in [t, 1[$ and $s \in]t - \delta, t[$,

$$\begin{aligned} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| &\leq \frac{1}{\tau} \left(|f(\frac{t}{\tau})| + |f(\frac{s}{\tau})| \right) |\psi(\tau)| \\ &\leq \frac{\|\varphi\|_p \|\psi\|_q}{\tau} \left((\ln \tau - \ln t)^{p-1} + (\ln \tau - \ln s)^{p-1} \right) (-\ln \tau)^{q-1}. \end{aligned}$$

Now there are two cases. If $p \leq 1$ we have that the function $s \mapsto (\ln \tau - \ln s)^{p-1}$ is nondecreasing, hence

$$(\ln \tau - \ln s)^{p-1} \leq (\ln \tau - \ln t)^{p-1} \quad (\tau \in [t, 1[, s \in]0, t]).$$

This means that, in this case,

$$\frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| \leq \frac{2\|\varphi\|_p \|\psi\|_q}{\tau} (\ln \tau - \ln t)^{p-1} (-\ln \tau)^{q-1} \quad (\tau \in [t, 1[, s \in]0, t]).$$

Moreover, the right hand side is integrable with respect to τ because, with the substitution $\tau = t^\sigma$, it follows that

$$(54) \quad \int_t^1 \frac{1}{\tau} (\ln \tau - \ln t)^{p-1} (-\ln \tau)^{q-1} d\tau = (-\ln t)^{p+q-1} B(p, q) < \infty.$$

When $p > 1$, then the function $s \mapsto (\ln \tau - \ln s)^{p-1}$ is decreasing, hence

$$(\ln \tau - \ln s)^{p-1} \leq (\ln \tau - \ln(t - \delta))^{p-1} \leq (-\ln(t - \delta))^{p-1} \quad (\tau \in [t, 1[, s \in]t - \delta, t]).$$

Thus, in this case, for all $\tau \in [t, 1[, s \in]t - \delta, t[$,

$$\frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| \leq \frac{\|\varphi\|_p \|\psi\|_q}{\tau} \left((\ln \tau - \ln t)^{p-1} + (-\ln(t - \delta))^{p-1} \right) (-\ln \tau)^{q-1}.$$

Again, the majorant is integrable because (54) holds, and (substituting $\tau = t^\sigma$)

$$\begin{aligned} \int_t^1 \frac{1}{\tau} (-\ln(t - \delta))^{p-1} (-\ln \tau)^{q-1} d\tau &= (-\ln(t - \delta))^{p-1} \int_t^1 \frac{1}{\tau} (-\ln \tau)^{q-1} d\tau \\ &= (-\ln(t - \delta))^{p-1} (-\ln t)^q B(1, q) < \infty. \end{aligned}$$

Therefore, Lebesgue's Theorem can be applied and hence (53) holds. Thus there exists $\delta^* \in]0, \delta]$, such that, for all $s \in]t - \delta^*, t[$,

$$(55) \quad \int_{]t, 1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{6}.$$

Combining the inequalities (47), (48), (49), (51), (52), and (55), we get

$$|(\varphi * \psi)(t) - (\varphi * \psi)(s)| < \varepsilon \quad (s \in]t - \delta^*, t[),$$

which proves the left-continuity of $\varphi * \psi$ at t .

To prove the right-continuity of $\varphi * \psi$ at t , we apply (47) for $s \in]t, t + \delta[$. The second inequality in (46) implies that

$$\text{meas} \left(]\frac{\ln s}{\ln t}, 1[\right) \leq \frac{\rho}{1 + \rho} < \rho.$$

Thus, inequality (43) holds with the interval $T :=]\frac{\ln s}{\ln t}, 1[$. Therefore, for the first term on the right hand side of (47) (using the estimates (42), and substituting $\tau = t^\sigma$ in the evaluation of the integral), we get

$$(56) \quad \int_t^s \frac{1}{\tau} |\varphi(\frac{t}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{6}.$$

To obtain an estimate for the second term on the right hand side of (47) (when $t < s$), we use $\varphi(x) = f(x)$ for $x \in H$ and obtain

$$(57) \quad \begin{aligned} \int_s^1 \frac{1}{\tau} |\varphi(\frac{t}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau &\leq \int_s^1 \frac{1}{\tau} \left(|\varphi(\frac{t}{\tau}) - f(\frac{t}{\tau})| + |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| + |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| \right) |\psi(\tau)| d\tau \\ &\leq \int_{]s,1[\setminus]sH^{-1}} \frac{1}{\tau} |\varphi(\frac{t}{\tau}) - f(\frac{t}{\tau})| |\psi(\tau)| d\tau + \int_{]s,1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau + \int_{]s,1[\setminus]sH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau. \end{aligned}$$

Applying an analogous argument as before, for $s \in]t, t + \delta[$, we can obtain the estimates

$$(58) \quad \int_{]s,1[\setminus]sH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{3} \quad \text{and} \quad \int_{]s,1[\setminus]tH^{-1}} \frac{1}{\tau} |f(\frac{s}{\tau}) - \varphi(\frac{s}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{3}.$$

Consider the second expression on the right hand side of (57). We will prove that

$$(59) \quad \lim_{s \rightarrow t+0} \int_{]s,1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau = 0.$$

First, with the substitution $\tau = \sigma^{\frac{\ln s}{\ln t}}$, for $s \in]t, t + \delta[$, we can obtain

$$(60) \quad \begin{aligned} \int_{]s,1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau &= \frac{\ln s}{\ln t} \int_{]t,1[} \frac{1}{\sigma} |f(t\sigma^{-\frac{\ln s}{\ln t}}) - f(s\sigma^{-\frac{\ln s}{\ln t}})| |\psi(\sigma^{\frac{\ln s}{\ln t}})| d\sigma \\ &\leq \int_{]t,1[} \frac{1}{\sigma} |f(t\sigma^{-\frac{\ln s}{\ln t}}) - f(s\sigma^{-\frac{\ln s}{\ln t}})| |\psi(\sigma^{\frac{\ln s}{\ln t}})| d\sigma. \end{aligned}$$

By the continuity of f and the local boundedness of ψ (which is a consequence of the inequality (42)), the integrand on the right hand side of (60) pointwise converges to zero as $s \rightarrow t+0$, hence, in view of Lebesgue's dominated convergence theorem, it suffices to show that the integrand admits an integrable majorant which is independent of $s \in]t, t + \delta[$. Using the inequality (42) and (45), we get that, for all $\tau \in]t, 1[$ and $s \in]t, t + \delta[$,

$$\begin{aligned} \frac{1}{\sigma} |f(t\sigma^{-\frac{\ln s}{\ln t}}) - f(s\sigma^{-\frac{\ln s}{\ln t}})| |\psi(\sigma^{\frac{\ln s}{\ln t}})| \\ \leq \|\varphi\|_p \|\psi\|_q \frac{1}{\sigma} \left((-\ln(t\sigma^{-\frac{\ln s}{\ln t}}))^{p-1} + (-\ln(s\sigma^{-\frac{\ln s}{\ln t}}))^{p-1} \right) (-\ln(\sigma^{\frac{\ln s}{\ln t}}))^{q-1}. \end{aligned}$$

Since $s \mapsto (-\ln s)^{q-1}$ is monotone on $]0, 1[$, therefore, for $s \in]t, t + \delta[$, we have

$$(-\ln(\sigma^{\frac{\ln s}{\ln t}}))^{q-1} = \left(\frac{-\ln \sigma \ln s}{\ln t} \right)^{q-1} \leq \max \left\{ 1, \left(\frac{\ln(t+\delta)}{\ln t} \right)^{q-1} \right\} (-\ln \sigma)^{q-1}.$$

Similarly, for $s \in]t, t + \delta[$ and $\sigma \in]t, 1[$, we get

$$(-\ln(s\sigma^{-\frac{\ln s}{\ln t}}))^{p-1} = \left(\frac{\ln s}{\ln t} \right)^{p-1} (-\ln t + \ln \sigma)^{p-1} \leq \begin{cases} \left(\frac{\ln(t+\delta)}{\ln t} \right)^{p-1} (-\ln t + \ln \sigma)^{p-1} & \text{if } p \leq 1, \\ (-\ln t + \ln \sigma)^{p-1} & \text{if } p > 1, \end{cases}$$

and

$$(-\ln(t\sigma^{-\frac{\ln s}{\ln t}}))^{p-1} = (-\ln t + \frac{\ln s \ln \sigma}{\ln t})^{p-1} \leq \begin{cases} (-\ln t + \ln \sigma)^{p-1} & \text{if } p \leq 1, \\ (-\ln t)^{p-1} & \text{if } p > 1. \end{cases}$$

Combining these inequalities, for $s \in]t, t + \delta[$ and $\sigma \in]t, 1[$, we obtain

$$\begin{aligned} & \frac{1}{\sigma} |f(t\sigma^{-\frac{\ln s}{\ln t}}) - f(s\sigma^{-\frac{\ln s}{\ln t}})| |\psi(\sigma^{\frac{\ln s}{\ln t}})| \\ & \leq \begin{cases} \|\varphi\|_p \|\psi\|_q (1 + (\frac{\ln(t+\delta)}{\ln t})^{p-1}) \max \{1, (\frac{\ln(t+\delta)}{\ln t})^{q-1}\} \frac{1}{\sigma} (-\ln t + \ln \sigma)^{p-1} (-\ln \sigma)^{q-1} & \text{if } p \leq 1, \\ \|\varphi\|_p \|\psi\|_q \max \{1, (\frac{\ln(t+\delta)}{\ln t})^{q-1}\} \frac{1}{\sigma} ((-\ln t + \ln \sigma)^{p-1} + (-\ln t)^{p-1}) (-\ln \sigma)^{q-1} & \text{if } p > 1. \end{cases} \end{aligned}$$

It is easy to check (by substituting $\sigma = t^\tau$) that the function on the right hand side of this inequality is integrable with respect to σ over $]t, 1[$. Therefore, Lebesgue's Theorem can be applied and hence (59) holds. Thus there exists $\delta^{**} \in]0, \delta^*]$, such that, for all $s \in]t, t + \delta^{**}[$,

$$(61) \quad \int_{]s, 1[} \frac{1}{\tau} |f(\frac{t}{\tau}) - f(\frac{s}{\tau})| |\psi(\tau)| d\tau < \frac{\varepsilon}{6}.$$

By summing up the respective sides of the inequalities (56), (57), (58), and (61), for all $s \in]t, t + \delta^{**}[$, we get

$$|(\varphi * \psi)(t) - (\varphi * \psi)(s)| < \varepsilon,$$

which completes the proof of the right-continuity of $\varphi * \psi$ at t .

To prove (39), let $t \in]0, 1]$ be fixed. Using (42) and substituting $\tau = t^s$, we get

$$\begin{aligned} |(\varphi * \psi)(t)| & \leq \int_t^1 \frac{1}{\tau} |\varphi(\frac{t}{\tau})| |\psi(\tau)| d\tau \leq \|\varphi\|_p \|\psi\|_q \int_t^1 \frac{1}{\tau} (-\ln(\frac{t}{\tau}))^{p-1} (-\ln \tau)^{q-1} d\tau \\ & = \|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1} \int_0^1 (1-s)^{p-1} s^{q-1} ds \\ & = B(p, q) \|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \|\varphi\|_p \|\psi\|_q (-\ln t)^{p+q-1}, \end{aligned}$$

which proves the inclusion $\varphi * \psi \in \Phi_{p+q}$ and the inequality (39).

In the proof of (40) first we use Fubini's theorem and then the variable t/τ is replaced by s :

$$\begin{aligned} \int_0^1 (\varphi * \psi)(t) dt & = \int_0^1 \int_t^1 \frac{1}{\tau} \varphi(\frac{t}{\tau}) \psi(\tau) d\tau dt = \int_0^1 \int_0^\tau \frac{1}{\tau} \varphi(\frac{t}{\tau}) \psi(\tau) dt d\tau \\ & = \int_0^1 \psi(\tau) \left(\frac{1}{\tau} \int_0^\tau \varphi(\frac{t}{\tau}) dt \right) d\tau = \int_0^1 \psi(\tau) d\tau \int_0^1 \varphi(s) ds. \end{aligned}$$

□

Lemma 3.3. *Let $p > 0$ be arbitrarily fixed, then, for all $x \in \mathbb{R}$,*

$$(62) \quad \lim_{n \rightarrow \infty} \frac{x^n}{\Gamma(np)} = 0$$

and the convergence is uniform on every compact interval of \mathbb{R} .

Proof. To prove the lemma, we will show that the series $\sum \frac{x^n}{\Gamma(np)}$ is convergent on \mathbb{R} . Using Cauchy's root test on this series and the Stirling formula for the Γ function, we get that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{|x|^n}{\Gamma(np)}} = \lim_{n \rightarrow \infty} \frac{|x|}{\sqrt[n]{\Gamma(np)}} = |x| \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{\frac{2\pi}{np} \left(\frac{np}{e}\right)^{np}}} = |x| \lim_{n \rightarrow \infty} \frac{e^p \sqrt[n]{np}}{\sqrt[n]{2\pi} (np)^p} = 0.$$

This means that the series is absolute convergent on \mathbb{R} and hence the convergence is uniform on compact subsets of \mathbb{R} , which yields the statement. □

Proposition 3.4. *Let $p > 0$ and $\varphi \in \Phi_p$. Define the sequence $\varphi_n :]0, 1[\rightarrow \mathbb{R}$ by the recursion*

$$(63) \quad \varphi_1 := \varphi, \quad \varphi_{n+1} := \varphi * \varphi_n \quad (n \in \mathbb{N}).$$

Then, for all $n \in \mathbb{N}$,

$$(64) \quad \varphi_n \in \Phi_{np}, \quad \|\varphi_n\|_{np} \leq \frac{(\Gamma(p))^n}{\Gamma(np)} \|\varphi\|_p^n, \quad \int_0^1 \varphi_n = \left(\int_0^1 \varphi \right)^n$$

and, for all $s \in]0, 1[$,

$$(65) \quad \lim_{n \rightarrow \infty} \varphi_n(s) = 0$$

furthermore, for all $\delta \in]0, 1[$, the convergence is uniform on $[\delta, 1[$.

Proof. By the definition of φ_1 , (64) holds trivially for $n = 1$. Assume that (64) is valid for some n . By Proposition 3.2 and the inductive assumption, $\varphi_{n+1} = \varphi * \varphi_n \in \Phi_{p+n p}$,

$$\|\varphi_{n+1}\|_{(n+1)p} \leq \frac{\Gamma(p)\Gamma(np)}{\Gamma((n+1)p)} \|\varphi\|_p \|\varphi_n\|_{np} \leq \frac{\Gamma(p)\Gamma(np)}{\Gamma((n+1)p)} \|\varphi\|_p \frac{(\Gamma(p))^n}{\Gamma(np)} \|\varphi\|_p^n = \frac{(\Gamma(p))^{n+1}}{\Gamma((n+1)p)} \|\varphi\|_p^{n+1},$$

and

$$\int_0^1 \varphi_{n+1} = \int_0^1 \varphi * \varphi_n = \int_0^1 \varphi \int_0^1 \varphi_n = \int_0^1 \varphi \left(\int_0^1 \varphi \right)^n = \left(\int_0^1 \varphi \right)^{n+1},$$

which proves (64) for $n + 1$. To prove (65), choose n_0 such that $n_0 p \geq 2$ and let $\delta \in]0, 1[$. Since the Gamma function is increasing on the interval $[2, \infty[$, by (64), for all $n \geq n_0$ and $s \in [\delta, 1[$, we have

$$(66) \quad \begin{aligned} |\varphi_{n+n_0}(s)| &\leq \|\varphi_{n+n_0}\|_{(n+n_0)p} (-\ln s)^{(n+n_0)p-1} \leq \frac{(\Gamma(p))^{n+n_0}}{\Gamma((n+n_0)p)} \|\varphi\|_p^{n+n_0} (-\ln s)^{(n+n_0)p-1} \\ &= (-\ln s)^{n_0 p-1} \Gamma(p)^{n_0} \|\varphi\|_p^{n_0} \frac{(\Gamma(p)(-\ln s)^p \|\varphi\|_p)^n}{\Gamma((n+n_0)p)} \\ &\leq (-\ln s)^{n_0 p-1} \Gamma(p)^{n_0} \|\varphi\|_p^{n_0} \frac{(\Gamma(p)(-\ln s)^p \|\varphi\|_p)^n}{\Gamma(np)} \\ &\leq (-\ln \delta)^{n_0 p-1} \Gamma(p)^{n_0} \|\varphi\|_p^{n_0} \frac{(\Gamma(p)(-\ln \delta)^p \|\varphi\|_p)^n}{\Gamma(np)}. \end{aligned}$$

Letting $n \rightarrow \infty$ in (66) and using Lemma 3.3, we get (65). The estimate (66) also ensures the uniformity of the convergence on the compact subsets of $]0, 1[$. \square

Proposition 3.5. *Let $g : [0, 1] \rightarrow \mathbb{R}$ be an upper bounded measurable function which is upper semicontinuous at 0. Let $p > 0$, $\varphi \in \Phi_p$ be a nonnegative function with $\int_0^1 \varphi = 1$ and define the sequence $\varphi_n :]0, 1[\rightarrow \mathbb{R}$ by (63). Then*

$$(67) \quad \limsup_{n \rightarrow \infty} \int_0^1 g(s) \varphi_n(s) ds \leq g(0).$$

Proof. Let $\varepsilon > 0$. If g is upper semicontinuous at 0, then there exists $\delta \in]0, 1[$, such that

$$(68) \quad g(s) < g(0) + \frac{\varepsilon}{2} \quad \text{for all } s \in]0, \delta[$$

and, by the upper boundedness, there exists $K > \max(0, -g(0))$, such that

$$(69) \quad g(s) \leq K \quad (s \in [0, 1]).$$

By Proposition 3.4, $\lim_{n \rightarrow \infty} \varphi_n = 0$ and the convergence is uniform also on $[\delta, 1[$. Hence, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$(70) \quad \varphi_n(s) \leq \frac{\varepsilon}{4K} \quad (s \in [\delta, 1]).$$

Using the third expression (64) and $\int_0^1 \varphi = 1$, we have that

$$(71) \quad \int_0^1 \varphi_n = \left(\int_0^1 \varphi \right)^n = 1.$$

Applying (71) and the nonnegativity of φ_n , we obtain

$$(72) \quad \int_0^1 g(s) \varphi_n(s) ds - g(0) = \int_0^\delta (g(s) - g(0)) \varphi_n(s) ds + \int_\delta^1 (g(s) - g(0)) \varphi_n(s) ds$$

To obtain an estimate for the first term of the right hand side, we use (68) and (71). Then, for all $n \in \mathbb{N}$,

$$(73) \quad \int_0^\delta (g(s) - g(0)) \varphi_n(s) ds \leq \int_0^\delta \frac{\varepsilon}{2} \varphi_n(s) ds \leq \frac{\varepsilon}{2} \int_0^1 \varphi_n(s) ds = \frac{\varepsilon}{2}.$$

Consider the second expression on the right hand side of (72). Then using (69) and (70), we obtain, for all $n \geq n_0$,

$$(74) \quad \int_\delta^1 (g(s) - g(0)) \varphi_n(s) ds \leq \int_\delta^1 2K \varphi_n(s) ds \leq \int_\delta^1 2K \frac{\varepsilon}{4K} ds = \frac{\varepsilon}{2} (1 - \delta) < \frac{\varepsilon}{2}.$$

Combining the inequalities (72), (73) and (74), we get that, for all $n \geq n_0$,

$$\int_0^1 g(s) \varphi_n(s) ds - g(0) < \varepsilon,$$

which proves the statement. \square

The proof of Theorem 1.3 is based on a sequence of lemmas.

Lemma 3.6. *Let $\alpha_H : D^* \rightarrow \mathbb{R}$ be even, $\rho : [0, 1] \rightarrow \mathbb{R}$ be integrable and $\lambda \in]0, 1[$. Then every $f : D \rightarrow \mathbb{R}$ lower hemicontinuous function satisfying the approximate Hermite–Hadamard inequality (7), fulfills*

$$(75) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \frac{\rho\left(\frac{1+s}{2}\right) + \rho\left(\frac{1-s}{2}\right)}{2} ds \\ \leq f(x) + f(y) + 2\alpha_H(x - y) \quad (x, y \in D).$$

Proof. Changing the role of x and y in (7), then adding the respective sides of the inequality so obtained and the original inequality (7), by the evenness of α_H , we get that

$$(76) \quad \int_0^1 (f(tx + (1-t)y) + f((1-t)x + ty)) \rho(t) dt \leq f(x) + f(y) + 2\alpha_H(x - y) \quad (x, y \in D).$$

Replacing t by $1 - t$ in the integral on the left hand side of (76), it follows that

$$(77) \quad \int_0^1 (f((1-t)x + ty) + f(tx + (1-t)y)) \rho(1-t) dt \leq f(x) + f(y) + 2\alpha_H(x - y) \quad (x, y \in D),$$

hence, adding the respective sides of the inequalities (76) and (77), we obtain

$$(78) \quad \int_0^1 (f(tx + (1-t)y) + f((1-t)x + ty)) \frac{\rho(t) + \rho(1-t)}{2} dt \leq f(x) + f(y) + 2\alpha_H(x - y) \quad (x, y \in D).$$

Finally, substituting $t := \frac{1+s}{2}$ in the integral on the left hand side of (78), we arrive at

$$(79) \quad \frac{1}{2} \int_{-1}^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \frac{\rho\left(\frac{1+s}{2}\right) + \rho\left(\frac{1-s}{2}\right)}{2} ds \\ \leq f(x) + f(y) + 2\alpha_H(x - y) \quad (x, y \in D).$$

Since the integrand on the left hand side of (79) is even, this inequality reduces to (75), which completes the proof of the lemma. \square

In what follows, we examine the Hermite-Hadamard inequality (75).

Lemma 3.7. *Let $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \geq 0$ and $p > 0$ such that (14) holds. Define $\varphi :]0, 1[\rightarrow \mathbb{R}$ by*

$$(80) \quad \varphi(s) := \frac{\rho\left(\frac{1+s}{2}\right) + \rho\left(\frac{1-s}{2}\right)}{2} \quad (s \in]0, 1[).$$

Then $\varphi \in \Phi_p$ and $\int_0^1 \varphi = 1$.

Proof. Inequality (14) results that

$$\varphi(s) = \frac{\rho\left(\frac{1+s}{2}\right) + \rho\left(\frac{1-s}{2}\right)}{2} \leq c(-\ln s)^{p-1} \quad (s \in]0, 1[),$$

which proves $\varphi \in \Phi_p$. The equality $\int_0^1 \varphi = 1$ is an automatic consequence of the assumption $\int_0^1 \rho = 1$. \square

Lemma 3.8. *Let $p, q > 0$ and $\varphi \in \Phi_p$, $\psi \in \Phi_q$ be nonnegative functions. Let $\alpha : D^* \rightarrow \mathbb{R}$ and let $\beta : D^* \rightarrow \mathbb{R}$ be a radially upper semicontinuous function. Assume that a lower hemicontinuous function $f : D \rightarrow \mathbb{R}$ satisfies the approximate Hermite-Hadamard inequalities*

$$(81) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \varphi(s) ds \leq f(x) + f(y) + 2\alpha(x - y) \quad (x, y \in D),$$

and

$$(82) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \psi(s) ds \leq f(x) + f(y) + 2\beta(x - y) \quad (x, y \in D).$$

Then f also satisfies the inequality

$$(83) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) (\varphi * \psi)(s) ds \\ \leq f(x) + f(y) + 2\alpha(x - y) + 2 \int_0^1 \beta(t(x - y)) \varphi(t) dt \quad (x, y \in D).$$

Proof. Assume that $f : D \rightarrow \mathbb{R}$ satisfies the inequalities (81) and (82). To prove (83), let $x, y \in D$ be fixed. Applying (82) for the elements $\frac{1+t}{2}x + \frac{1-t}{2}y, \frac{1-t}{2}x + \frac{1+t}{2}y \in D$, we obtain

$$\int_0^1 \left(f\left(\frac{1+ts}{2}x + \frac{1-ts}{2}y\right) + f\left(\frac{1-ts}{2}x + \frac{1+ts}{2}y\right) \right) \psi(s) ds \\ \leq f\left(\frac{1+t}{2}x + \frac{1-t}{2}y\right) + f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) + 2\beta(t(x - y)).$$

Multiplying this inequality by $\varphi(t)$ and integrating the functions on both sides with respect to t on $]0, 1[$, then using that f also satisfies (81), we get that

$$\begin{aligned}
 & \int_0^1 \int_0^1 \left(f\left(\frac{1+ts}{2}x + \frac{1-ts}{2}y\right) + f\left(\frac{1-ts}{2}x + \frac{1+ts}{2}y\right) \right) \psi(s) ds \varphi(t) dt \\
 (84) \quad & \leq \int_0^1 \left(f\left(\frac{1+t}{2}x + \frac{1-t}{2}y\right) + f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \right) \varphi(t) dt + 2 \int_0^1 \beta(t(x-y)) \varphi(t) dt \\
 & \leq f(x) + f(y) + 2\alpha(x-y) + 2 \int_0^1 \beta(t(x-y)) \varphi(t) dt.
 \end{aligned}$$

Now we compute the left hand side of the previous inequality. Substituting $s = \frac{\tau}{t}$ and using also Fubini's theorem, we obtain

$$\begin{aligned}
 & \int_0^1 \int_0^1 \left(f\left(\frac{1+ts}{2}x + \frac{1-ts}{2}y\right) + f\left(\frac{1-ts}{2}x + \frac{1+ts}{2}y\right) \right) \psi(s) ds \varphi(t) dt \\
 (85) \quad & = \int_0^1 \int_0^t \left(f\left(\frac{1+\tau}{2}x + \frac{1-\tau}{2}y\right) + f\left(\frac{1-\tau}{2}x + \frac{1+\tau}{2}y\right) \right) \psi\left(\frac{\tau}{t}\right) \frac{1}{t} d\tau \varphi(t) dt \\
 & = \int_0^1 \left(\left(f\left(\frac{1+\tau}{2}x + \frac{1-\tau}{2}y\right) + f\left(\frac{1-\tau}{2}x + \frac{1+\tau}{2}y\right) \right) \int_\tau^1 \frac{1}{t} \psi\left(\frac{\tau}{t}\right) \varphi(t) dt \right) d\tau \\
 & = \int_0^1 \left(f\left(\frac{1+\tau}{2}x + \frac{1-\tau}{2}y\right) + f\left(\frac{1-\tau}{2}x + \frac{1+\tau}{2}y\right) \right) (\psi * \varphi)(\tau) d\tau
 \end{aligned}$$

. Combining (84) and (85), the inequality (83) follows, which completes the proof. \square

Lemma 3.9. *Let $\varphi :]0, 1[\rightarrow \mathbb{R}_+$ be an integrable function and let $\beta : D^* \rightarrow \mathbb{R}$ be a radially upper semicontinuous. Then, the function $\gamma : D^* \rightarrow \mathbb{R}$ defined by*

$$(86) \quad \gamma(u) := \int_0^1 \beta(tu) \varphi(t) dt \quad (u \in D^*)$$

is also radially upper semicontinuous on D^ .*

Proof. To prove that γ defined by (86) is radially upper semicontinuous at $u_0 \in D^*$, let $s_n \rightarrow s_0$ be an arbitrary sequence in $[0, 1]$. We have that

$$\beta(ts_n u_0) \leq \sup_{\tau \in [0, 1]} \beta(\tau u_0) =: K \quad (t \in [0, 1], n \in \mathbb{N}),$$

thus, $K\varphi$ is an integrable majorant for the sequence of functions $t \mapsto \beta(ts_n u_0)\varphi(t)$. Using Fatou's lemma and the radial upper semicontinuity of β , we get that

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \gamma(s_n u_0) &= \limsup_{n \rightarrow \infty} \int_0^1 \beta(ts_n u_0) \varphi(t) dt \\
 &\leq \int_0^1 \limsup_{n \rightarrow \infty} \beta(ts_n u_0) \varphi(t) dt \leq \int_0^1 \beta(ts_0 u_0) \varphi(t) dt = \gamma(s_0 u_0),
 \end{aligned}$$

which proves the statement. \square

Lemma 3.10. *Let $p > 0$, $\varphi \in \Phi_p$ be a nonnegative and $\alpha_H : D^* \rightarrow \mathbb{R}$ be a radially upper semicontinuous function. If $f : D \rightarrow \mathbb{R}$ is lower hemicontinuous and fulfills the approximate Hermite–Hadamard inequality*

$$(87) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \varphi(s) ds \leq f(x) + f(y) + 2\alpha_H(x-y) \quad (x, y \in D),$$

then, for all $n \in \mathbb{N}$, the function f also satisfies the Hermite–Hadamard inequality

$$(88) \quad \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \varphi_n(s) ds \leq f(x) + f(y) + 2\alpha_n(x-y) \quad (x, y \in D),$$

where the sequences $\varphi_n :]0, 1[\rightarrow \mathbb{R}_+$ and $\alpha_n : D^* \rightarrow \mathbb{R}_+$ are defined by (63) and

$$(89) \quad \alpha_1 = \alpha_H, \quad \alpha_{n+1}(u) = \int_0^1 \alpha_n(tu) \varphi(t) dt + \alpha_H(u) \quad (u \in D^*),$$

respectively.

Proof. We note that, by Lemma 3.9, the sequence of functions (α_n) is well-defined and α_n is radially lower semicontinuous for all $n \in \mathbb{N}$.

Let $x, y \in D$ and $p > 0$. To prove (88), we use induction on $n \in \mathbb{N}$. For $n = 1$, we have (87). Assume that (88) holds for $n \in \mathbb{N}$. Since $\varphi \in \Phi_p$, by Proposition 3.4, we have that $\varphi_n \in \Phi_{np}$. The function f satisfies (87) and also (88), for $n \in \mathbb{N}$. Thus, in Lemma 3.8, (81) holds with the functions φ and $\alpha := \alpha_1$. Furthermore, by the inductive assumption, also in Lemma 3.8, (82) holds with the functions $\psi := \varphi_n$ and $\beta := \alpha_n$, for $n \in \mathbb{N}$. Hence the function f also fulfills the Hermite–Hadamard inequality (83), which results,

$$\begin{aligned} & \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) (\varphi * \varphi_n)(s) ds \\ & \leq f(x) + f(y) + 2\alpha_H(x-y) + 2 \int_0^1 \alpha_n(t(x-y)) \varphi(t) dt, \end{aligned}$$

which is the case $n + 1$. □

Lemma 3.11. *Let $p > 0$ and $\varphi \in \Phi_p$ be a nonnegative function with $\int_0^1 \varphi(t) dt = 1$ and $f : D \rightarrow \mathbb{R}$ be lower hemicontinuous. Then*

$$(90) \quad \liminf_{n \rightarrow \infty} \int_0^1 \left(f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \right) \varphi_n(s) ds \geq 2f\left(\frac{x+y}{2}\right) \quad (x, y \in D).$$

Proof. To prove the statement, let $x, y \in D$ and $p > 0$. Define $g_{x,y} : [0, 1] \rightarrow \mathbb{R}$ by

$$g_{x,y}(s) := f\left(\frac{1+s}{2}x + \frac{1-s}{2}y\right) + f\left(\frac{1-s}{2}x + \frac{1+s}{2}y\right) \quad (s \in [0, 1]).$$

The lower hemicontinuity of f implies that $g_{x,y}$ is lower semicontinuous and hence lower bounded on $[0, 1]$. Thus, we can apply Proposition 3.5, for $g := -g_{x,y}$ and $\varphi \in \Phi_p$, which yields that

$$\liminf_{n \rightarrow \infty} \int_0^1 g_{x,y}(s) \varphi_n(s) ds \geq g_{x,y}(0).$$

This inequality is equivalent to (90). □

Lemma 3.12. *Let $p > 0$ and $\varphi \in \Phi_p$ be a nonnegative function, and $\alpha_H : D^* \rightarrow \mathbb{R}$ be a radially upper semicontinuous function. Then, for all $n \in \mathbb{N}$, the function $\alpha_n : D^* \rightarrow \mathbb{R}$ defined by (89) is radially upper semicontinuous and the sequence (α_n) is nondecreasing [nonincreasing], whenever α_H is nonnegative [nonpositive]. Furthermore, if $\alpha_J : D^* \rightarrow \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality*

$$(91) \quad \alpha_J(u) \geq \int_0^1 \alpha_J(su) \varphi(s) ds + \alpha_H(u) \quad (u \in D^*),$$

then

$$(92) \quad \limsup_{n \rightarrow \infty} \alpha_n(u) \leq \alpha_J(u) - \alpha_J(0) + \alpha_H(0) \quad (u \in D^*).$$

Proof. The statement about the radial upper semicontinuity directly follows from Lemma 3.9.

Assume first that α_H is nonnegative. We will prove by induction on $n \in \mathbb{N}$, that the sequence (α_n) is nondecreasing, i.e.,

$$(93) \quad \alpha_{n+1} \geq \alpha_n \quad (n \in \mathbb{N}).$$

For $n = 1$, by the nonnegativity of $\alpha_1 = \alpha_H$, we have that

$$\alpha_2(u) = \int_0^1 \alpha_1(su)\varphi(s)ds + \alpha_H(u) \geq \alpha_1(u) \quad (u \in D^*).$$

Assume that (93) holds for some $n \in \mathbb{N}$ and consider the case $n + 1$. Using the definition of α_{n+1} , the inductive assumption and the nonnegativity of α_n , we get that

$$\alpha_{n+2}(u) = \int_0^1 \alpha_{n+1}(su)\varphi(s)ds + \alpha_H(u) \geq \int_0^1 \alpha_n(su)\varphi(s)ds + \alpha_H(u) = \alpha_{n+1}(u) \quad (u \in D^*).$$

Analogously, if α_H is nonpositive, we can obtain that the sequence (α_n) is nonincreasing.

To prove (92), let $\alpha_J : D^* \rightarrow \mathbb{R}$ be a radially lower semicontinuous solution of (91). Subtracting the respective sides of the inequalities (91) from (89), for the sequence of functions $g_n := \alpha_n - \alpha_J$, we obtain

$$g_{n+1}(u) \leq \int_0^1 g_n(su)\varphi(s)ds \quad (u \in D^*, n \in \mathbb{N}),$$

We obviously have that g_n is also radially upper semicontinuous.

Iterating this inequality, similarly as in Lemma 3.8 and Lemma 3.10, it can be proved that

$$(94) \quad g_{n+1}(u) \leq \int_0^1 g_1(su)\varphi_n(s)ds \quad (u \in D^*, n \in \mathbb{N}),$$

where φ_n is defined by (63). Taking the limsup as $n \rightarrow \infty$ in (94), by Proposition 3.5, we get that

$$\limsup_{n \rightarrow \infty} g_{n+1}(u) \leq \limsup_{n \rightarrow \infty} \int_0^1 g_1(su)\varphi_n(s)ds \leq g_1(0) = \alpha_H(0) - \alpha_J(0) \quad (u \in D^*),$$

which immediately yields (92). \square

Proof of Theorem 1.3. Assume that the conditions of Theorem 1.3 hold and $f : D \rightarrow \mathbb{R}$ is an upper semicontinuous solution of (7). Then by Lemma 3.6, f also fulfills (87), where $\varphi :]0, 1[\rightarrow \mathbb{R}_+$ is defined by (80). Then, by Lemma 3.7, $\varphi \in \Phi_p$ and $\int_0^1 \varphi = 1$. Using Lemma 3.10, we get that (88) also holds, where, for all $n \in \mathbb{N}$, $\alpha_n : D^* \rightarrow \mathbb{R}$ is defined by (89). Since α_J satisfies the functional inequality (15), thus applying Fubini's theorem, then substituting $s := 1 - 2t$ and $s := 2t - 1$, we get that

$$\begin{aligned} \alpha_J(u) &\geq \int_0^{1/2} \alpha_J((1 - 2t)u)\rho(t)dt + \int_{1/2}^1 \alpha_J((2t - 1)u)\rho(t)dt + \alpha_H(u) \\ &= \int_0^1 \alpha_J(su)\rho\left(\frac{1-s}{2}\right)\frac{1}{2}ds + \int_0^1 \alpha_J(su)\rho\left(\frac{1+s}{2}\right)\frac{1}{2}ds + \alpha_H(u) \\ &= \int_0^1 \alpha_J(su)\frac{\rho\left(\frac{1+s}{2}\right) + \rho\left(\frac{1-s}{2}\right)}{2}ds + \alpha_H(u) = \int_0^1 \alpha_J(su)\varphi(s)ds + \alpha_H(u), \end{aligned}$$

which means that (91) also holds. Taking the liminf as $n \rightarrow \infty$ in (88), using also Lemma 3.11, Lemma 3.12 and $\alpha_H(0) \leq \alpha_J(0)$, we get that

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) &\leq f(x) + f(y) + 2 \liminf_{n \rightarrow \infty} \alpha_n(x-y) \leq f(x) + f(y) + 2 \limsup_{n \rightarrow \infty} \alpha_n(x-y) \\ &\leq f(x) + f(y) + 2(\alpha_J(x-y) + \alpha_H(0) - \alpha_J(0)) \leq f(x) + f(y) + 2\alpha_J(x-y). \end{aligned}$$

Hence (5) holds, which completes the proof of Theorem 1.3. \square

In what follows, we examine the case, when X is a normed space and α_H is a linear combination of the powers of the norm with positive exponents, i.e., if α_H is of the form

$$(95) \quad \alpha_H(u) := \int_{]0, \infty[} \|u\|^q d\mu_H(q) \quad (u \in D^*),$$

where μ_H is a signed Borel measure on the interval $]0, \infty[$. An important particular case is when μ_H is of the form $\sum_{i=1}^k c_i \delta_{q_i}$, where $c_1, \dots, c_k \in \mathbb{R}$, $q_1, \dots, q_k > 0$ and δ_q denotes the Dirac measure concentrated at q .

Theorem 3.13. *Let $\rho : [0, 1] \rightarrow \mathbb{R}$ be integrable with $\int_0^1 \rho = 1$ and assume that there exist $c \geq 0$ and $p > 0$ such that (14) holds. Let $\lambda \in [0, 1]$ and μ_H be a signed Borel measure on $]0, \infty[$ such that*

$$(96) \quad \int_{]0, \infty[} \|u\|^q d|\mu_H|(q) < \infty \quad (u \in D^*)$$

and

$$(97) \quad \int_{]0, \infty[} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} d|\mu_H|(q) < \infty.$$

Assume that $f : D \rightarrow \mathbb{R}$ is lower hemicontinuous and satisfies the Hermite-Hadamard type inequality

$$(98) \quad \int_0^1 f(tx + (1-t)y) \rho(t) dt \leq \lambda f(x) + (1-\lambda)f(y) + \int_{]0, \infty[} \|x-y\|^q d\mu_H(q) \quad (x, y \in D).$$

Then f also fulfils the Jensen type inequality

$$(99) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + \int_{]0, \infty[} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} \|x-y\|^q d\mu_H(q) \quad (x, y \in D).$$

Proof. By Theorem 1.3, it suffices to show that the function

$$\alpha_J(u) := \int_{]0, \infty[} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} \|u\|^q d\mu_H(q) \quad (u \in D^*)$$

is well-defined and satisfies (15) with equality where $\alpha_H : D^* \rightarrow \mathbb{R}$ is defined by

$$\alpha_H(u) := \int_{]0, \infty[} \|u\|^q d\mu_H(q) \quad (u \in D^*)$$

To see that, for all $u \in D^*$, $\alpha_J(u)$ is finite, we distinguish two cases. If $\|u\| \leq 1$, then $\|u\|^q \leq 1$ for all $q > 0$, and hence, by assumption (97),

$$|\alpha_J(u)| \leq \int_{]0, \infty[} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} d|\mu_H|(q) < \infty.$$

Now let $\|u\| > 1$. Then, the functions $q \mapsto \|u\|^q$ and $q \mapsto \int_0^1 (1 - |1 - 2t|^q) \rho(t) dt$ are increasing functions, hence

$$\begin{aligned} |\alpha_J(u)| &\leq \|u\| \int_{]0, 1]} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} d|\mu_H|(q) + \left(\int_0^1 (1 - |1 - 2t|) \rho(t) dt \right)^{-1} \int_{]1, \infty[} \|u\|^q d|\mu_H|(q), \end{aligned}$$

which is again finite by conditions (96) and (97).

To prove that α_J satisfies (15), using $\int_0^1 \rho = 1$, we compute

$$\begin{aligned} &\int_0^1 \alpha_J(|1 - 2s|u) \rho(s) ds + \alpha_H(u) \\ &= \int_0^1 \int_{]0, \infty[} \left(\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt \right)^{-1} \|1 - 2s|u\|^q d\mu_H(q) \rho(s) ds + \int_{]0, \infty[} \|u\|^q d\mu_H(q) \\ &= \int_{]0, \infty[} \left(\frac{\int_0^1 |1 - 2s|^q \rho(s) ds}{\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt} + 1 \right) \|u\|^q d\mu_H(q) = \int_{]0, \infty[} \frac{\|u\|^q}{\int_0^1 (1 - |1 - 2t|^q) \rho(t) dt} d\mu_H(q) = \alpha_J(u), \end{aligned}$$

which proves that (15) holds with equality. \square

Corollary 3.14. *Let $\lambda \in [0, 1]$, $a \in \mathbb{R}$ and $q > 0$. Assume that $f : D \rightarrow \mathbb{R}$ is lower hemicontinuous and satisfies the Hermite–Hadamard type inequality*

$$\int_0^1 f(tx + (1-t)y) dt \leq \lambda f(x) + (1-\lambda)f(y) + a\|x - y\|^q \quad (x, y \in D).$$

Then f also fulfils the Jensen type inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + a \frac{q+1}{q} \|x - y\|^q \quad (x, y \in D).$$

Proof. Observe that the constant weight function $\rho \equiv 1$ satisfies the assumptions of Theorem 3.13 with $c = p = 1$. Also, with $\mu_H := a\delta_q$, conditions (96) and (97) hold trivially. Thus, the conclusion of Theorem 3.13 is valid with

$$\begin{aligned} \alpha_J(u) &= \frac{a\|u\|^q}{\int_0^1 (1 - |1 - 2t|^q) dt} = \frac{a\|u\|^q}{\int_0^{\frac{1}{2}} (1 - (1 - 2t)^q) dt + \int_{\frac{1}{2}}^1 (1 - (2t - 1)^q) dt} \\ &= \frac{a\|u\|^q}{2\left(\frac{1}{2} - \frac{1}{2(q+1)}\right)} = a \frac{q+1}{q} \|u\|^q \quad (u \in D^*), \end{aligned}$$

which proves the statement. \square

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